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An algebraic approach to quadratic parametric processes

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Abstract. A Lie algebraic approach to quadratic parametric processes in quantum optics and quantum acoustics is presented. In this approach the Heisenberg-Weyl and symplectic dynamical algebras are used to obtain a general exact solution for the time evolution operator. The solution is then applied to describe quantum mechanically the parametric process of backward-wave echo generation in dielectrics.

1. Introduction

The most general bilinear Hamiltonian, describing quadratic parametric processes, is given by

$$\mathscr{H}(t)=\sum_{i,j=1}^{2}\omega_{ij}(t)a_i^+a_j+[b_{ij}(t)a_ia_j+d_i(t)a_i]+\text{HC} \qquad (1)$$

where ω, b and d are arbitrary complex valued functions of time and the annihilation and creation operators a_i, a_j^+ satisfy the commutation relations $[a_i, a_j^+]=\delta_{ij}$, $i, j=1, 2$. We use a quantum mechanical model for quadratic parametric processes, treating an external field classically and neglecting the losses. The Hamiltonian (1) involves the following as special cases:

(i) parametric generation and amplification (Louisell *et al* 1961, Mollow and Glauber 1967) ($\omega_{12}=\omega_{21}=0$, $b_{11}=b_{22}=0$, $d_i=0$);

(ii) frequency conversion (Louisell *et al* 1961, Mollow and Glauber 1967) ($b_{ij}=0$, $d_i=0$);

(iii) degenerate parametric amplification (Raiford 1974) ($\omega_{12}=\omega_{21}=\omega_{22}=0$, $b_{12}=b_{21}=b_{22}=0$, $d_i=0$);

(iv) second harmonic generation in two modes (Bloembergen 1965) ($\omega_{12}=\omega_{21}=0$, $b_{12}=b_{21}=0$, $d_i=0$);

(v) generation of squeezed states (Yuen 1976) and a quadrupole oscillator echo process (Kopvillem and Prants 1985) ($\omega_{12}=\omega_{21}=\omega_{22}=0$, $b_{12}=b_{21}=b_{22}=0$);

(vi) parametric backward-wave echo generation (Fedders and Lu 1973, Kopvillem and Prants 1985) ($\omega_{12}=\omega_{21}=0$, $b_{11}=b_{22}=0$) and

(vii) parametric backward-wave echo generation with frequency conversion ($b_{ij}=0$).

In general, the dynamical behaviour of a quantum system is described by the time evolution operator $U(t, t_0)$ which connects the Schrödinger and Heisenberg pictures of the motion. It allows us to find the time development of the dynamical operators in the Heisenberg picture along with the certain transition probabilities in the Schrödinger picture and the temporal evolution of the density matrix.

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In this paper a Lie algebraic approach (§ 2) is used to calculate explicitly the time evolution operator for quadratic parametric processes with Hamiltonian (1). Constructing the appropriate dynamical Lie algebra (§ 3), we factorise the operator $U(t, t_0)$ into the product of the exponentials of the generators of the algebra. We derive a general exact solution for $U(t, t_0)$ (§ 4) that may be used to treat all the quadratic parametric processes mentioned above. Special attention will be paid to the quantum theory of backward-wave echoes (§ 5).

2. Lie algebraic solution of the Schrödinger equation

If the Hamiltonian, governing a quantum system $\mathscr{H}(t)=\Sigma_{j=1}^{m} h_j(t)H_j$ (H_j are constant operators), generates a finite-dimensional Lie algebra L_n: $\{H_1, H_2, \ldots, H_n\}$, $n \geq m$, then the time-dependent Schrödinger equation

$$\mathrm{i}\hbar\,\partial U(t, t_0)/\partial t=\mathscr{H}(t)U(t, t_0) \qquad U(t, t_0)=I \tag{2}$$

has an exact solution of the form

$$U(t, t_0)=\prod_{j=1}^{n} \exp[g_j(t, t_0)H_j] \tag{3}$$

where the complex valued functions of time $g_j(t, t_0)$ obey the equation (Wei and Norman 1963)

$$(\mathrm{i}\hbar)^{-1}\sum_{j=1}^{n} h_j(t)H_j=\sum_{j=1}^{n}\frac{\partial g_j(t, t_0)}{\partial t}\left(\prod_{k=1}^{j-1}\exp[g_k(t, t_0)\ \mathrm{Ad}\ H_k]\right)H_j \tag{4}$$

where $(\exp \mathrm{Ad}\ H_k)H_j \equiv (\exp g_k H_k)H_j \exp(-g_k H_k)$.

The decomposition $L=S\oplus R$ of L into the semi-direct sum of the semi-simple subalgebra S and radical R gives rise to the corresponding decomposition $\mathscr{H}=\mathscr{H}_S+\mathscr{H}_R$ of $\mathscr{H}$ that gives rise, in turn, to the solution $U=U_S\cdot U_R$ with the factors U_S and U_R obeying the equations

$$\mathrm{i}\hbar\,\partial U_S/\partial t=\mathscr{H}_S(t)U_S \qquad \mathrm{i}\hbar\,\partial U_R/\partial t=U_S^{+}\mathscr{H}_R(t)U_S U_R. \tag{5}$$

Now, functions $g_k^S(t, t_0)$ and $g_l^R(t, t_0)$ satisfy two equations similar to equation (4) but much more simple. The derivations of Wei–Norman equations (4) and (5) are given in appendix 1.

For discussions of Lie groups and Lie algebras see, for example, the books of Hamermesh (1962) and Gilmore (1974).

3. Dynamical algebras for quadratic parametric processes

The dynamical group for quadratic parametric processes is the semi-direct product of a four-dimensional symplectic group Sp(4, R) and the Heisenberg–Weyl group N(2). The corresponding dynamical algebra is the semi-direct sum; sp(4, R)$\oplus$n(2) of the semi-simple subalgebra sp(4, R) and the radical n(2). The Lie algebra sp(4, R) is realised by the set of all bilinear products formed from a_k and a_l^+. One can show that the ten operators $\{a_k a_l, a_k^+ a_l, a_k^+ a_l^+\}$ close under the symplectic algebra sp(4, R). The five generators of n(2) can be written as $\{a_k, a_l^+, I\}$, where $k, l=1, 2$ and I is the identity operator.

Individual quadratic parametric processes mentioned above have the following dynamical algebras†:

(i) $u(1,1)=\{a_1^+a_1+\frac{1}{2}, a_2^+a_2+\frac{1}{2}, a_1^+a_2^+, a_1a_2\}$
(ii) $u(2)=\{a_1^+a_1+\frac{1}{2}, a_2^+a_2+\frac{1}{2}, a_1^+a_2, a_1a_2^+\}$
(iii) $su(1,1)=\{a_1^+a_1+\frac{1}{2}, a_1^2, a_1^{+2}\}$
(iv) $su(1,1)\oplus su(1,1)$
(v) $su(1,1)\oplus n(1)$
(vi) $u(1,1)\oplus n(2)$
(vii) $u(2)\oplus n(2)$.

Our algebraic approach reduces the solution of a Schrödinger equation (2) with any quadratic Hamiltonian (1) to the solution of equation (2) with the Hamiltonian of the process (vi) which generates the algebra $u(1,1)\oplus n(2)$. The reason is that the dynamical algebras for processes (i), (iii), (iv) and (v) are the subalgebras of $u(1,1)\oplus n(2)$ and the algebra $u(2)$ for the processes (ii) and (vii) is connected with $u(1,1)$ by the Weyl unitary trick. In other words, one-to-one mapping $a_1\leftrightarrow a_1'$, $a_1^+\leftrightarrow(a_1')^+$, $a_2\leftrightarrow -i(a_2')^+$, $a_2^+\leftrightarrow -ia_2'$ provides an isomorphism (equivalence) between algebras $u(1,1)$ and $u(2)$ (primed operators belong to $u(2)$). In fact, we deal with the same commutation relations. Carrying out the direct transformations $a_1'\rightarrow a_1,\ldots$ in equations (2)–(4), we obtain the corresponding equations for $u(1,1)$. The time evolution operator for $u(2)$ may be found from the $u(1,1)$ solutions (7) and (9) with the help of the inverse transformations $a_1\rightarrow a_1',\ldots$. The explicit calculation is given in appendix 2.

4. Time evolution operator for $u(1,1)\oplus n(2)$

Now we calculate the time evolution operator for the backward-wave echo process (vi) whose Hamiltonian

$$\mathscr{H}=\hbar\omega_1(a_1^+a_1+\tfrac{1}{2})+\hbar\omega_2(a_2^+a_2+\tfrac{1}{2})+d_0(t)I$$
$$+\{-\hbar\gamma\exp[i(\omega_1+\omega_2)t]a_1a_2+d_1(t)a_1+d_2(t)a_2\}+\mathrm{HC} \tag{6}$$

generates the dynamical algebra $u(1,1)\oplus n(2)$. Using equations (3) and (5), we are able to factorise the time evolution operator into the product of exponential operators $U=U_{u(1,1)}\cdot U_{n(2)}$, where

$$U_{u(1,1)}=\exp[g_1(a_1^+a_1+\tfrac{1}{2})]\exp[g_2(a_2^+a_2+\tfrac{1}{2})]\exp[g_3a_1a_2]\exp[g_4a_1^+a_2^+] \tag{7}$$

$$U_{n(2)}=\exp f_1a_1\exp f_2a_1^+\exp f_3a_2\exp f_4a_2^+\exp f_5I. \tag{8}$$

The time-dependent functions g obey equation (4) which, in the case $u(1,1)$, is equivalent to a set of four differential equations with the following solutions:

$$g_1=-i\omega_1t+\ln\cosh\gamma t \qquad g_2=-i\omega_2t+\ln\cosh\gamma t$$
$$g_3=\tfrac{1}{2}i\sinh 2\gamma t \qquad g_4=i\tanh\gamma t. \tag{9}$$

The functions f are found from equation (4) which can easily be solved by quadrature

† It is possible to realise all the dynamical algebras by other combinations of boson operators, for example, $su(1,1)=\{-\frac{1}{4}(a^2+a^{+2}), -\frac{1}{4}i(a^2-a^{+2}), \frac{1}{4}(aa^++a^+a)\}$.

for the solvable algebra n(2):

$$f_1 = \mathrm{i}\hbar \int_{t_0}^{t} (d_1 \, \mathrm{e}^{g_1} - g_3 d_2^* \, \mathrm{e}^{-g_2}) \, \mathrm{d}t \qquad f_3 = \mathrm{i}\hbar \int_{t_0}^{t} (d_2 \, \mathrm{e}^{g_2} - g_3 d_1^* \, \mathrm{e}^{-g_1}) \, \mathrm{d}t$$

$$f_2 = \mathrm{i}\hbar \int_{t_0}^{t} (d_1^* \, \mathrm{e}^{-g_1} + d_2 g_4 \, \mathrm{e}^{g_2} - g_3 g_4 d_1^* \, \mathrm{e}^{-g_1}) \, \mathrm{d}t \qquad f_5 = \int_{t_0}^{t} \left(\mathrm{i}\hbar d_0 - \frac{\partial f_2}{\partial t} f_1 - \frac{\partial f_4}{\partial t} f_3 \right) \mathrm{d}t$$

$$f_4 = \mathrm{i}\hbar \int_{t_0}^{t} (d_2^* \, \mathrm{e}^{-g_2} + d_1 g_4 \, \mathrm{e}^{g_1} - g_3 g_4 d_2^* \, \mathrm{e}^{-g_2}) \, \mathrm{d}t. \tag{10}$$

5. Backward-wave echoes

Here we would like to apply the algebraic approach, developed in the preceding sections, to a quantum mechanical description of a parametric backward-wave echo generation in piezoelectrics. The phenomenon has been observed in a number of different materials (Frenois *et al* 1973, Fossheim and Holt 1982, Smolyakov and Schtirkov 1976). In a typical experiment, the first ultrasonic pulse of frequency ω and wavevector $\boldsymbol{k}$ is excited piezoelectrically at the left end of a sample. At $t = \tau$, a second electric field pulse of frequency 2ω, a peak amplitude E_0, width Δt and wavevector $\boldsymbol{k}_e$ is applied. Parametric coupling between the forward travelling acoustic wave generated by the first pulse and the microwave electric field of the second pulse results in the generation of a backward travelling acoustic wave of frequency ω and wavevector $-\boldsymbol{k}$ and in the possible amplification of the forward travelling wave. The backward travelling wave is detected as an echo signal at the left end of the sample at $t = 2\tau$. In quantum mechanical terms, the backward-wave echo is generated as a consequence of the annihilation of a photon of frequency 2ω and wavevector $\boldsymbol{k}_e$, which is small compared to $\boldsymbol{k}$ in the region of ultrasonic frequencies, and the creation of two phonons $(\omega, \boldsymbol{k})$ and $(\omega, -\boldsymbol{k})$ (Bajak 1977). While the phonons $(\omega, -\boldsymbol{k})$ create a backward travelling wave the phonons $(\omega, \boldsymbol{k})$ contribute to the forward travelling wave which appears as an amplification.

Following Kopvillem and Prants (1982) all echo phenomena can be separated into one of two classes according to the microscopic physical nature of the particles or objects that possess phase memory and generate the echo signal. In backward-wave multipole echo processes the objects are fixed electroacoustical multipoles connected with the structural defects or impurities in crystals. In backward-wave phonon echo processes the objects are phonons. While the n-pulse echo experiments $(n \geqslant 2)$ can be described as an echo of multipoles, only the two-pulse experiments can be described as an echo of phonons. Both types of echoes can be generated in piezoelectrics under the experimental conditions mentioned above.

Backward-wave dipole echo is generated by the structural electroacoustical dipoles which can radiate both the forward travelling phonons (a_1, a_1^+) and the backward travelling phonons (a_2, a_2^+). The Hamiltonian (6) of the process (with $d_2(t) = 0$) generates the dynamical algebra $\mathrm{u}(1,1) \oplus \mathrm{n}(2)$ with the term $d_1 a_1 + d_1^* a_1^+$ describing the dipole–phonon interaction of the first pulse and the term $\hbar\gamma \exp[\mathrm{i}(\omega_1 + \omega_2)t] a_1 a_2 +$ HC describing the non-linear electroacoustical interaction of the second pulse. The time evolution operator $U_{\mathrm{u}(1,1)} \cdot U_{\mathrm{n}(2)}$ for such a process was calculated in § 4, and the intensity of the echo signal could be calculated in the usual way (Kopvillem and Prants 1982, 1985). Up to now, there are no reliable experimental facts on the backward-wave dipole echoes. That is why we prefer to calculate explicitly the intensity of the backward-wave phonon echoes in piezoelectrics, which have been detected in a number of experiments, as an example of the application of the algebraic approach.

If the pump electric field of the second pulse is treated classically, the total Hamiltonian of the quadratic parametric process, which was discussed at the beginning of this section,

$$\mathscr{H}_{\mathrm{u}(1,1)} = \hbar\omega(a_1^+a_1+\tfrac{1}{2}) + \hbar\omega(a_2^+a_2+\tfrac{1}{2}) - \{\hbar\omega\gamma E_0 \exp[-\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}+\boldsymbol{k}_e)\boldsymbol{r}]a_1a_2\} + \mathrm{HC} \qquad (11)$$

generates the dynamical algebra u(1, 1). Here $a_1^+(a_1)$ and $a_2^+(a_2)$ are the phonon creation (annihilation) operators for the forward and backward travelling phonons, respectively. The coupling constant has the form $\gamma = (2\sqrt{2}\rho v_k v_{-k})^{-1}\partial c/\partial E$, where $\partial c/\partial E$ is the non-linear piezoelectric coefficient which is obtained from Taylor's expansion of the potential energy of the crystal (Bajak 1977), ρ is the density of the material, v_k and v_{-k} are corresponding sound velocities and c is the elastic constant of the crystal.

With $U_{\mathrm{u}(1,1)}$ given by (7) and (9), the number of backward travelling phonons is

$$\langle a_2^+a_2\rangle = \mathrm{Tr}\ U^+_{\mathrm{u}(1,1)}a_2^+(0)a_2(0)U_{\mathrm{u}(1,1)}\rho_{a_1}(0) = (\bar{n}_{a_1}+1)\sinh^2(\gamma E_0\omega\Delta t) \qquad (12)$$

where $\bar{n}_{a_1}$ is the average number of phonons a_1 in the forward travelling mode at $t=0$ and the initial density matrix is given by

$$\rho_{a_1}(0) = \exp[-\hbar\omega(a_1^+a_1+\tfrac{1}{2})/k_BT]\{\mathrm{Tr}\exp[-\hbar\omega(a_1^+a_1+\tfrac{1}{2})/k_BT]\}^{-1}.$$

In a recent ω-2ω experiment (Meredith *et al* 1984) on $LiNbO_3$ the observed variation of the backward-wave echo amplitude agrees well with $\sinh\Theta$, where Θ is proportional to the amplitude of the pump electric field, E_0. The initial ultrasonic pulse was excited piezoelectrically at the left end of a sample at a frequency of 17.3 GHz. The right end was placed in a microwave electric field region of a rectangular cavity, resonating at 34.6 GHz.

6. Concluding remarks

The algebraic approach to quadratic parametric processes presented in this paper is based on a dynamical algebra sp(4, R)$\oplus$n(2). The most general bilinear Hamiltonian for such processes can be expressed in terms of the generators of the algebra. It allows us to derive an exact solution for the time evolution operator which contains more complete information on the dynamical behaviour of a quantum system than the Heisenberg equations usually used to describe the parametric processes. Such a solution is applied to describe the parametric generation of backward-wave echo in piezoelectrics. The algebraic method allows for the calculation of the corresponding intensity in non-perturbative fashion. The recent experimental data agree well with our theoretical result. Our approach is general enough in that the various quadratic processes can be described, grouped and classified by their dynamical algebras.

Appendix 1. Derivations of equations (4) and (5)

Let U be of the form (3). Then we may write

$$\frac{\partial U}{\partial t} \equiv \frac{\partial g_1}{\partial t}\mathrm{e}^{g_1H_1}H_1\prod_{k=2}^{n}\mathrm{e}^{g_kH_k}+\ldots+\frac{\partial g_n}{\partial t}\prod_{k=1}^{n}\mathrm{e}^{g_kH_k}H_n \equiv \sum_{j=1}^{n}\frac{\partial g_j}{\partial t}\left(\prod_{k=1}^{j-1}\mathrm{e}^{g_kH_k}\right)H_j\left(\prod_{k=j}^{n}\mathrm{e}^{g_kH_k}\right).$$

Substituting the derivative in equation (2) and right-multiplying the result by the operator U^{-1}, one can obtain

$$(\mathrm{i}\hbar)^{-1}\sum_{j=1}^{n} h_j H_j = \sum_{j=1}^{n}\frac{\partial g_j}{\partial t}\left(\prod_{k=1}^{j-1} \mathrm{e}^{g_k H_k}\right) H_j \left(\prod_{k=j-1}^{1} \mathrm{e}^{-g_k H_k}\right).$$

Let us define the operator Ad H_k and its powers in the usual way:

$$(\mathrm{Ad}\, H_k)H_j \equiv [H_k, H_j] \qquad (\mathrm{Ad}\, H_k)^2 H_j \equiv [H_k, [H_k, H_j]] \ldots$$

where H_k, H_j are the elements of the algebra L. Then $\mathrm{e}^{H_k} H_j\, \mathrm{e}^{-H_k} \equiv (\mathrm{e}^{\mathrm{Ad}\, H_k})H_j \in L$, and we may rewrite the preceding equation in the form of equation (4).

If $L = S \oplus R$, then $\mathscr{H} = \mathscr{H}_S + \mathscr{H}_R$, where $\mathscr{H}_S \in S$ and $\mathscr{H}_R \in R$. One can directly verify that the solution $U = U_S \cdot U_R$ satisfies the equation $\mathrm{i}\hbar\, \partial U/\partial t = (\mathscr{H}_S + \mathscr{H}_R)U$ if U_S and U_R obey the equations

$$\mathrm{i}\hbar\, \partial U_S/\partial t = \mathscr{H}_S U_S \qquad \mathrm{i}\hbar\, \partial U_R/\partial t = (U_S^+ \mathscr{H}_R U_S) U_R.$$

Since R is a radical in L, i.e. $[R, L] \in R$, we have $U_S^+ \mathscr{H}_R U_S \in R$.

Appendix 2. Calculation of the time evolution operator for u(2)

The u(2) Hamiltonian for the frequency conversion process

$$\mathscr{H}_{\mathrm{u}(2)} = \hbar\omega_1'(a_1'^+ a_1' + \tfrac{1}{2}) + \hbar\omega_2'(a_2'^+ a_2 + \tfrac{1}{2}) + \hbar\gamma'\{\exp[\mathrm{i}(\omega_1' - \omega_2')t] a_1' a_2'^+\} + \mathrm{HC}$$

can be transformed to the u(1, 1) form

$$\mathscr{H}_{\mathrm{u}(1,1)} = \hbar\omega_1(a_1^+ a_1 + \tfrac{1}{2}) + \hbar\omega_2(a_2^+ a_2 + \tfrac{1}{2}) - \hbar\gamma\{\exp[\mathrm{i}(\omega_1 + \omega_2)] a_1 a_2\} + \mathrm{HC}$$

by means of the Weyl unitary trick $a_1' \to a_1$, $a_1'^+ \to a_1^+$, $a_2' \to -\mathrm{i}a_2^+$, $a_2'^+ \to -\mathrm{i}a_2$, where $\omega_1 \equiv \omega_1'$, $\omega_2 \equiv -\omega_2'$, $\gamma \equiv \mathrm{i}\gamma'$. Then the time evolution operator for u(2) can be obtained from the solutions (7) and (9) with the help of the inverse transformation

$$U_{\mathrm{u}(2)} = \exp[g_1'(a_1'^+ a_1 + \tfrac{1}{2})] \exp[-g_2'(a_2'^+ a_2' + \tfrac{1}{2})] \exp(\mathrm{i}g_3' a_1' a_2'^+) \exp(\mathrm{i}g_4' a_1'^+ a_2')$$

$$g_1' = -\mathrm{i}\omega_1' t + \ln\cos\gamma' t \qquad g_2' = \mathrm{i}\omega_2' t + \ln\cos\gamma' t$$

$$g_3' = -\tfrac{1}{2}\sin 2\gamma' t \qquad g_4' = -tg\gamma' t.$$

where we have used the well known identities $\cosh \mathrm{i}\,\gamma' t \equiv \cos\gamma' t$ and $\sinh \mathrm{i}\,\gamma' t \equiv \mathrm{i}\sin\gamma' t$.

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